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T-duality, fiber bundles and matrices

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Abstract: We extend the T-duality for gauge theory to that on curved space described as a nontrivial fiber bundle. We also present a new viewpoint concerning the consistent truncation and the T-duality for gauge theory and discuss the relation between the vacua on the total space and on the base space. As examples, we consider $S^3(/Z_k)$, $S^5(/Z_k)$ and the Heisenberg nilmanifold.

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1. Introduction

Emergence of space-time is one of the key concepts in nonperturbative definition of superstring or M-theory by matrix models [1, 2]. This phenomenon in field theory was found over two decades ago in the large N reduction of gauge theories [3–7], which states equivalence under some conditions between a large N gauge theory and the matrix model that is its dimensional reduction to a point. This equivalence originates from the fact that the eigenvalues of matrices can be interpreted as momenta. This interpretation reappeared in the T-duality between the low-energy effective theories for Dp-branes and for D$(p-1)$-branes [8, 9]. More concretely, this T-duality tells that a $\mathrm{U}(N)$ gauge theory on $R^p \times S^1$ is equivalent to the $\mathrm{U}(N \times \infty)$ gauge theory that is its dimensional reduction to R^p if a periodicity condition is imposed to the theory on R^p.

The main purpose of this paper is to extend the T-duality for gauge theory to that on curved space described as a nontrivial fiber bundle. The above mentioned T-duality is concerning a trivial S^1 bundle, $R^p \times S^1$. We restrict ourselves to principal S^1 bundles and show the T-duality between the gauge theories on the total space and on the base space. We also present a new viewpoint concerning the consistent truncation and the T-duality for gauge theory. Furthermore, we discuss the properties of the vacua[1] on the total space and the base space. In our previous publication [10] on the gauge/gravity correspondence for the SU(2|4) symmetric theories [11] (see also [12−17]), we showed the T-duality between $\mathcal{N} = 4$ super Yang Mills (SYM) on $R \times S^3(/Z_k)$ and $2+1$ SYM on $R \times S^2$, which is suggested from the gravity side. This is regarded as the T-duality on $S^3(/Z_k)$, which is a nontrivial S^1 fibration over S^2. In this paper, we generalize this result. Our findings would be useful for the study of describing curved space-time in matrix models [18−20] as well as the study of curved D-branes.

This paper is organized as follows. In section 2, we review the T-duality for gauge theory in a standard way. In section 3, we present a new viewpoint concerning the consistent truncation and the T-duality for gauge theory. Although this viewpoint is not necessarily needed for the proof of the T-duality on fiber bundle, it is interesting itself and indeed makes the T-duality for gauge theory more plausible. In section 4, we consider a dimensional reduction from the total space of a principal S^1 bundle to its base space. In section 5, we show the T-duality between the gauge theories on the base space and the total space. In section 6, we discuss the properties of the nontrivial vacua on the total space and the base space. We classify the vacua on the total space and discuss their relation to the vacua on the base space. In section 7, we present some examples: $S^3(/Z_k)$, $S^5(/Z_k)$ and the Heisenberg nilmanifold. Section 8 is devoted to conclusion and discussion.

2. Review of T-duality for gauge theory

In this section, we give a standard review of the T-duality between the gauge theories on $R^p \times S^1$ and R^p [8, 9]. We first consider pure Yang Mills on $R^p \times S^1$:

$$S_{p+1} = \frac{1}{g_{p+1}^2} \int d^{p+1}z \, \frac{1}{4} \mathrm{Tr}(F_{MN}F_{MN}), \tag{2.1}$$

where z^M $(M = 1, \ldots, p+1)$ are decomposed into (x^μ, y) $(\mu = 1, \ldots, p)$, y parameterizes S^1 with the radius R and $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$. By putting $A_\mu = a_\mu,\ A_y = \phi$ and dropping the y-dependence, this theory is dimensionally reduced to Yang Mills with a Higgs field ϕ on R^p:

$$S_p = \frac{1}{g_p^2} \int d^p x \, \mathrm{Tr} \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{2} D_\mu \phi D_\mu \phi \right), \tag{2.2}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$, $D_\mu \phi = \partial_\mu \phi - i[a_\mu, \phi]$ and $g_p^2 = \frac{1}{2\pi R} g_{p+1}^2$. Here, we adopt $\mathrm{U}(N \times \infty)$ as the gauge group of S_p. Namely, the fields in S_p are hermitian matrices consisting of infinitely many blocks, each of which is an $N \times N$ matrix. We label the blocks by (s, t), where s, t run from $-\infty$ to ∞. Then, S_p is expressed in terms of the blocks as

[1] Throughout this paper, we consider gauge theories on Riemannian manifolds with a positive-definite metric. In the following arguments, we can easily add the time direction as direct product. To be precise, the 'vacua' in this paper mean the classical vacua of the corresponding gauge theories on this direct product space.

follows:

$$S_p = \frac{1}{g_p^2}\int d^p x \sum_{s,t} \mathrm{tr}\left(\frac{1}{4} f_{\mu\nu}^{(s,t)} f_{\mu\nu}^{(t,s)} + \frac{1}{2}(D_\mu\phi)^{(s,t)}(D_\mu\phi)^{(t,s)}\right), \tag{2.3}$$

where tr stands for the trace over the $N \times N$ matrix.

We make an S^1 compactification with the radius $\tilde{R}$ in the ϕ direction by imposing the following conditions on the fields:

$$\begin{aligned} U a_\mu U^\dagger &= a_\mu, \\ U\phi U^\dagger &= \phi + 2\pi\tilde{R}\mathbf{1}_{N\times\infty}, \end{aligned} \tag{2.4}$$

where U is the 'shift' matrix with infinite size,

$$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & \mathbf{0}_N & \mathbf{1}_N & & & \\ & & \mathbf{0}_N & \mathbf{1}_N & & \\ & & & \mathbf{0}_N & \mathbf{1}_N & \\ & & & & \mathbf{0}_N & \ddots \\ & & & & & \ddots \end{pmatrix}. \tag{2.5}$$

These conditions are expressed in terms of the block components as

$$\begin{aligned} a_\mu^{(s+1,t+1)} &= a_\mu^{(s,t)}, \\ \phi^{(s+1,t+1)} &= \phi^{(s,t)} + 2\pi\tilde{R}\delta_{st}\mathbf{1}_N. \end{aligned} \tag{2.6}$$

They can be solved as

$$a_\mu = \hat{a}_\mu + \tilde{a}_\mu, \quad \phi = \hat{\phi} + \tilde{\phi} \tag{2.7}$$

with

$$\hat{a}_\mu = 0, \quad \hat{\phi} = 2\pi\tilde{R}\,\mathrm{diag}(\ldots, s-1, s, s+1, \ldots) \otimes \mathbf{1}_N \quad (\hat{\phi}^{(s,t)} = 2\pi\tilde{R}s\delta_{st}) \tag{2.8}$$

and

$$\tilde{a}_\mu^{(s,t)} = \tilde{a}_\mu^{(s-t)}, \quad \tilde{\phi}^{(s,t)} = \tilde{\phi}^{(s-t)}. \tag{2.9}$$

The background (2.8) is a vacuum of (2.2). The fluctuations around the vacuum, $\tilde{a}_\mu^{(s,t)}$ and $\tilde{\phi}^{(s,t)}$, depend only on $s-t$ as indicated in (2.9), which represents a periodicity. The above procedure should be called orbifolding.

By making the Fourier transformation, which turns out to be interpreted as the T-duality, one can recover pure Yang Mills on $R^p \times S^1$, where the radius of the original S^1 is R. The fields on $R^p \times S^1$ are defined in terms of the fields on R^p as

$$\begin{aligned} A_\mu(x,y) &= \sum_w \tilde{a}_\mu^{(w)}(x) e^{-\frac{i}{R}wy}, \\ A_y(x,y) &= \sum_w \tilde{\phi}^{(w)}(x) e^{-\frac{i}{R}wy}. \end{aligned} \tag{2.10}$$

The radius of the original S^1, R, is related to the radius of the dual S^1, $\tilde{R}$, as

$$R = \frac{1}{2\pi\tilde{R}}. \tag{2.11}$$

Then, the block components in (2.3) are evaluated as

$$\begin{aligned}
(D_\mu\phi(x))^{(s,t)} =& \partial_\mu\tilde{\phi}^{(s-t)}(x) + i2\pi\tilde{R}(s-t)\tilde{a}_\mu^{(s-t)}(x) \\
& - i\sum_u(\tilde{a}_\mu^{(s-u)}(x)\tilde{\phi}^{(u-t)}(x) - \tilde{\phi}^{(s-u)}(x)\tilde{a}_\mu^{(u-t)}(x)) \\
=& \frac{1}{2\pi R}\int_0^{2\pi R} dyF_{\mu y}(x,y)e^{\frac{i}{R}(s-t)y}, \\
f_{\mu\nu}^{(s,t)}(x) =& \frac{1}{2\pi R}\int_0^{2\pi R} dyF_{\mu\nu}(x,y)e^{\frac{i}{R}(s-t)y}.
\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.3) yields

$$S_p = \frac{1}{g_p^2}\frac{1}{2\pi R}\sum_w\int d^{p+1}z\,\frac{1}{4}\mathrm{tr}(F_{MN}F_{MN}). \tag{2.13}$$

By dividing the above expression by the overall factor $\sum_w$, which gives an infinite constant, one indeed reproduces the original pure Yang Mills on $R^p \times S^1$ (2.1) with the gauge group U(N).

In the context of the D-brane effective theories, the above procedure is interpreted as the T-duality between Dp-brane and D($p-1$)-brane, although the $9-p$ Higgs fields and the fermions are omitted here for simplicity. The background (2.8) represents an infinite array of stacks of N coincident D($p-1$)-branes, where 's' labels the s-th stack. The distance between the neighboring stacks is $2\pi\tilde{R}$. (2.9) expresses the periodicity which produces the dual S^1 with the radius $\tilde{R}$. $\tilde{a}_\mu^{(w)}$ and $\tilde{\phi}^{(w)}$ represent an open string stretched between the s-th stack and the $(s+w)$-th stack, so that $-w$ corresponds to the winding number around the dual S^1. In (2.10), the winding number $-w$ is reinterpreted as the momentum $-w/R$ along the original S^1 with the radius R. The relation between the radii (2.11) is the same as that for the T-duality in string theory. Dividing (2.13) by the overall factor $\sum_w$ corresponds to extracting a single period. In this way, the effective theory for a stack of N coincident Dp-branes is obtained through the T-duality.

3. Consistent truncation and T-duality

In the previous section, we reviewed the T-duality for gauge theory in a standard way. In this section, we present a new viewpoint concerning the consistent truncation and the T-duality.

Let the gauge group in (2.1) be U(M). We consider a pure-gauge background,

$$\begin{aligned}
\hat{A}_\mu &= 0 = -i\partial_\mu VV^\dagger, \\
\hat{A}_y &= \frac{1}{R}\mathrm{diag}(\ldots,\underbrace{n_{s-1},\ldots,n_{s-1}}_{N_{s-1}},\underbrace{n_s,\ldots,n_s}_{N_s},\underbrace{n_{s+1},\ldots,n_{s+1}}_{N_{s+1}},\ldots) = -i\partial_y VV^\dagger,
\end{aligned} \tag{3.1}$$

with

$$V = \mathrm{diag}(\ldots, \underbrace{e^{\frac{i}{R}n_{s-1}y}, \ldots, e^{\frac{i}{R}n_{s-1}y}}_{N_{s-1}}, \underbrace{e^{\frac{i}{R}n_{s}y}, \ldots, e^{\frac{i}{R}n_{s}y}}_{N_{s}}, \underbrace{e^{\frac{i}{R}n_{s+1}y}, \ldots, e^{\frac{i}{R}n_{s+1}y}}_{N_{s+1}}, \ldots), \quad (3.2)$$

where $M = \ldots + N_{s-1} + N_s + N_{s+1} + \ldots$. Due to the single-valuedness of V, all n_s must be integers. We assume that all n_s are different. This background naturally induces a block structure for $M \times M$ matrices. We label the blocks by (s,t), where the (s,t) block is an $N_s \times N_t$ matrix.

We denote the fluctuations of A_M around the background (3.1) by $\tilde{A}_M$, while we continue to use A_M for the fields around the trivial background $A_M = 0$. Since the background (3.1) is gauge-equivalent to the trivial background, we have a relation

$$A_M = -i\partial_M V^\dagger V + V^\dagger(\hat{A}_M + \tilde{A}_M)V, \quad (3.3)$$

which is equivalent to

$$A_M = V^\dagger \tilde{A}_M V. \quad (3.4)$$

For the (s,t) block, (3.4) implies

$$A_M^{(s,t)} = e^{-\frac{i}{R}(n_s - n_t)y}\tilde{A}_M^{(s,t)}. \quad (3.5)$$

We make the Fourier expansions for $A_M^{(s,t)}$ and $\tilde{A}_M^{(s,t)}$ with respect to the y direction as

$$\begin{aligned} A_M^{(s,t)}(x,y) &= \sum_m A_{M,m}^{(s,t)}(x) e^{\frac{i}{R}my}, \\ \tilde{A}_M^{(s,t)}(x,y) &= \sum_m \tilde{A}_{M,m}^{(s,t)}(x) e^{\frac{i}{R}my}. \end{aligned} \quad (3.6)$$

From (3.5), we obtain a relation between the Kaluza-Klein (KK) modes,

$$A_{M,m-(n_s-n_t)}^{(s,t)}(x) = \tilde{A}_{M,m}^{(s,t)}(x). \quad (3.7)$$

The theory around the trivial background of (2.1) is written in terms of $A_{M,m}^{(s,t)}$ while the theory around the background (3.1) of (2.1) in terms of $\tilde{A}_{M,m}^{(s,t)}$. The two theories are equivalent under the identification (3.7). The trivial background $A_M = 0$ corresponds to the trivial vacuum of the theory. Due to the variety of the choices of n_s and N_s, we have many different representations of the theory around the trivial vacuum.

In the usual KK reduction, one keeps only $A_{M,0}^{(s,t)}$ in the theory around the trivial background of (2.1). This is a consistent truncation, because the momentum 'm' is conserved, and one obtains the theory around the trivial vacuum $a_\mu = 0,\ \phi = 0$ of (2.2). Similarly, one can keep only $\tilde{A}_{M,0}^{(s,t)}$ in the theory around the background (3.1) of (2.1) to truncate (2.1) consistently. It is seen from (3.1) that the resulting theory is the theory around a vacuum of (2.2) given by

$$\hat{a}_\mu = 0, \quad \hat{\phi} = 2\pi\tilde{R}\mathrm{diag}(\ldots, \underbrace{n_{s-1}, \ldots, n_{s-1}}_{N_{s-1}}, \underbrace{n_s, \ldots, n_s}_{N_s}, \underbrace{n_{s+1}, \ldots, n_{s+1}}_{N_{s+1}}, \ldots). \quad (3.8)$$

In this theory, $\tilde{A}^{(s,t)}_{\mu,0}$ and $\tilde{A}^{(s,t)}_{y,0}$ are identified with $\tilde{a}^{(s,t)}_{\mu}$ and $\tilde{\phi}^{(s,t)}$, respectively. This theory is no longer equivalent to the theory around the trivial vacuum of (2.2), although these two theories originate from the same theory. In other words, we can obtain many different theories by consistently truncating the original theory in different ways. Indeed, (3.7) tells us that keeping only $A^{(s,t)}_{M,-(n_s-n_t)}$ in the theory around the trivial background of (2.1) yields the theory around the vacuum (3.8) of (2.2). That this is a consistent truncation can also be understood from the fact that the sum of the charge '$n_s - n_t$' and the momentum 'm' is conserved because so is each of them. Note that in the theory around the vacuum (3.8) of (2.2) the gauge symmetry $\mathrm{U}(M)$ is spontaneously broken to $\ldots \times \mathrm{U}(N_{s-1}) \times \mathrm{U}(N_s) \times \mathrm{U}(N_{s+1}) \times \ldots$.

By using the above discussions, we can easily show in an alternative way the T-duality reviewed in the previous section. Let us consider the case in which $M = N \times \infty$, s runs from $-\infty$ to ∞, $N_s = N$ for all s and $n_s = s$. In this case, the vacuum (3.8) is nothing but the vacuum (2.8) considered in the previous section. In the theory around the trivial background of (2.1), we impose the constraint

$$A^{(s,t)}_{M,m} = A^{(s-t)}_{M,m}, \tag{3.9}$$

and keep only $A^{(s-t)}_{M,-(s-t)}$. The summations over the block indices $s, t, \ldots$ are identified with the summations over the momenta. From the momentum conservation, we have the overall factor $\sum_w$. Thus we obtain the theory around the trivial vacuum of $\mathrm{U}(N)$ Yang Mills on $R^p \times S^1$ with the overall factor $\sum_w$, where $A^{(-m)}_{M,m}$ is identified with the KK mode $A_{M,m}$ of the $\mathrm{U}(N)$ theory. We see, therefore, from the discussion in the previous paragraph that the theory around the vacuum (2.8) of (2.2) with the periodicity condition (2.9) imposed is equivalent to the theory around the trivial vacuum of (2.1) with the gauge group $\mathrm{U}(N)$ and the overall factor $\sum_w$. This is indeed the T-duality reviewed in the previous section.

4. Dimensional reduction from total space to base space

In this section, we perform a dimensional reduction from the total space of a principal S^1 bundle to its base space. We consider a principal S^1 bundle whose total space is a $(D+1)$-dimensional manifold P and whose base space is a D-dimensional manifold B. The projection is given by $\pi : P \to B$. The base space B has a covering $\{U_{[I]}\}$ $(I = 1, 2, \ldots)$, each element of which is parameterized by $x^{\mu}_{[I]}$ $(\mu = 1, \ldots, D)$. The total space P has a covering $\{\pi^{-1}(U_{[I]})\}$. $\pi^{-1}(U_{[I]})$ is diffeomorphic to $U_{[I]} \times S^1$ by the local trivialization, so that it is parameterized by $z^M_{[I]} = (x^{\mu}_{[I]}, y_{[I]})$ $(M = 1, \ldots, D+1)$, where $y_{[I]} = z^{D+1}_{[I]}$ parameterizes the S^1 and $0 \leq y_{[I]} < 2\pi R$. If there is overlap between $U_{[I]}$ and $U_{[I']}$, the relation between $y_{[I]}$ and $y_{[I']}$ is determined by the transition function $e^{-\frac{i}{R}v_{[II']}}$ as $y_{[I']} = y_{[I]} - v_{[II']}(x_{[I]})$. In the following, we add a subscript or superscript $[I]$ to quantities which are evaluated on $U_{[I]}$. Quantities without such a subscript or superscript are independent of which patch is used to evaluate them.

We assume that the total space possesses the $\mathrm{U}(1)$ isometry in the fiber direction and the size of the fiber, namely the radius of S^1, is constant. The metrics that satisfy such conditions generally take the form on $\pi^{-1}(U_{[I]})$

$$ds^2_{D+1} = G^{[I]}_{MN} dz^M_{[I]} dz^N_{[I]} = g^{[I]}_{\mu\nu}(x_{[I]}) dx^\mu_{[I]} dx^\nu_{[I]} + (dy_{[I]} + b^{[I]}_\mu(x_{[I]}) dx^\mu_{[I]})^2, \tag{4.1}$$

where $b^{[I]} = b^{[I]}_\mu dx^\mu_{[I]}$ must be transformed as $b^{[I']} = b^{[I]} + dv_{[II']}$. From this metric, one can define a connection 1-form on the principal bundle as follows. First, note that connection 1-forms in general take the form

$$\omega = \frac{1}{R}(dy_{[I]} + t^{[I]}_\mu(x_{[I]}) dx^\mu_{[I]}), \tag{4.2}$$

where $t^{[I]}$ must be transformed as $t^{[I']} = t^{[I]} + dv_{[II']}$. Second, we introduce an orthonormal basis for the tangent space of the total space, $E_A \quad (A = 1, \dots, D+1)$, such that the direction of E_{D+1} coincides with the fiber direction. Explicitly, the elements of E_A are given by

$$\begin{aligned} E^{[I]\mu}_\alpha &= e^{[I]\mu}_\alpha, \\ E^{[I]y}_\alpha &= -e^{[I]\nu}_\alpha b^{[I]}_\nu = -b^{[I]}_\alpha, \\ E^{[I]\mu}_{D+1} &= 0, \\ E^{[I]y}_{D+1} &= 1, \end{aligned} \tag{4.3}$$

where $\alpha = 1, \dots, D$ and $e^{[I]\mu}_\alpha$ is determined from $g^{[I]\mu\nu} = e^{[I]\mu}_\alpha e^{[I]\nu}_\alpha$. E_α span the subspace orthogonal to the fiber direction. Then, ω is determined from the condition $\omega(E_\alpha) = 0$ for all α as

$$\omega = \frac{1}{R}(dy_{[I]} + b^{[I]}). \tag{4.4}$$

The orthonormal basis E^A of the cotangent space dual to (4.3) is given by

$$\begin{aligned} E^{[I]\alpha}_\mu &= e^{[I]\alpha}_\mu, \\ E^{[I]\alpha}_y &= 0, \\ E^{[I]D+1}_\mu &= b^{[I]}_\mu, \\ E^{[I]D+1}_y &= 1, \end{aligned} \tag{4.5}$$

which are identified with the vielbeins of the total space. The indices 'A' are the local Lorentz indices for the total space. One can identify the space spanned by E_α, in which the inner product is given by G in (4.1), with the tangent space of the base space with the same inner product. Then, it follows from (4.5) that $e^{[I]\alpha}_\mu$ are the vielbeins of the D-dimensional base space, namely $g^{[I]}_{\mu\nu}$ are the metric of the base space and α are the local Lorentz indices for the base space. Note that $\frac{1}{R} b^{[I]}_\mu$ gives a connection 1-form of the vector bundle associated with the principal bundle. The spin connections, $\Omega_A{}^B{}_C = E^{[I]M}_A \Omega^{[I]B}_M{}_C$, are determined from (4.5) as

$$\begin{aligned} \Omega_\alpha{}^\beta{}_\gamma &= \omega_\alpha{}^\beta{}_\gamma, \\ \Omega_\alpha{}^\beta{}_{D+1} &= \frac{1}{2} b_{\alpha\beta}, \\ \Omega_{D+1}{}^\alpha{}_\beta &= \frac{1}{2} b_{\beta\alpha}, \\ \Omega_{D+1}{}^\alpha{}_{D+1} &= 0, \end{aligned} \tag{4.6}$$

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where $\omega_\alpha{}^\beta{}_\gamma$ are the spin connections on the base space evaluated from $e_\mu^{[I]\alpha}$, and $b_{\alpha\beta} = \nabla_\alpha b_\beta^{[I]} - \nabla_\beta b_\alpha^{[I]}$.

We start with pure Yang Mills on the $(D+1)$-dimensional total space:

$$S_{D+1} = \frac{1}{g_{D+1}^2} \int d^{D+1}z \sqrt{G} \frac{1}{4} \mathrm{Tr}(F_{AB} F_{AB}), \tag{4.7}$$

where $d^{D+1}z \sqrt{G}$ represents the invariant volume. We dimensionally reduce this theory to Yang Mills with a Higgs field on the D-dimensional base space. Since we decomposed the (co)tangent space of the total space into the fiber direction and the directions orthogonal to it in (4.3) and (4.5), we naturally relate the gauge fields A_A on the total space to the gauge fields a_α and the Higgs field ϕ on the base space as follows:

$$\begin{aligned} A_\alpha &= a_\alpha, \\ A_{D+1} &= \phi, \end{aligned} \tag{4.8}$$

where we assume that the both sides in (4.8) are independent of $y_{[I]}$. By using (4.6), we evaluate the field strength on the total space as

$$\begin{aligned} F_{\alpha\beta} &= f_{\alpha\beta} + b_{\alpha\beta}, \\ F_{\alpha D+1} &= D_\alpha \phi, \end{aligned} \tag{4.9}$$

where $f_{\alpha\beta} = \nabla_\alpha a_\beta - \nabla_\beta a_\alpha - i[a_\alpha, a_\beta]$.

By using (4.9) and $\sqrt{G^{[I]}} = \sqrt{g^{[I]}}$, we obtain from (4.7) Yang Mills with the Higgs field ϕ on the base space:[2]

$$S_D = \frac{1}{g_D^2} \int d^D x \sqrt{g}\, \mathrm{Tr} \left(\frac{1}{4} (f_{\alpha\beta} + b_{\alpha\beta}\phi)(f_{\alpha\beta} + b_{\alpha\beta}\phi) + \frac{1}{2} D_\alpha \phi D_\alpha \phi \right), \tag{4.10}$$

where $g_D^2 = \frac{1}{2\pi R} g_{D+1}^2$. Note that there appears the $\mathrm{U}(1)$ curvature $b_{\alpha\beta}$ in (4.10)

5. T-duality on fiber bundle

The discussion on the consistent truncation of Yang Mills on the total space of the principal S^1 bundle proceeds parallel to that of Yang Mills on $R^p \times S^1$ in section 3. By using the discussion, we can show the T-duality between the gauge theories on the total space and on the base space. As before, let the gauge group in (4.7) be $\mathrm{U}(M)$. We consider a gauge transformation which is an analogue of V in (3.2). Such a gauge transformation should be defined locally on each patch. It is given on $\pi^{-1}(U_{[I]})$ by

$$\begin{aligned} V_{[I]} = & \\ & \mathrm{diag}(\ldots, \underbrace{e^{\frac{i}{R} n_{s-1} y_{[I]}}, \ldots, e^{\frac{i}{R} n_{s-1} y_{[I]}}}_{N_{s-1}}, \underbrace{e^{\frac{i}{R} n_s y_{[I]}}, \ldots, e^{\frac{i}{R} n_s y_{[I]}}}_{N_s}, \underbrace{e^{\frac{i}{R} n_{s+1} y_{[I]}}, \ldots, e^{\frac{i}{R} n_{s+1} y_{[I]}}}_{N_{s+1}}, \ldots), \end{aligned} \tag{5.1}$$

[2]This action is formally the same as that derived in [21], where the compactification of gravitational and Yang Mills system from a direct product space-time $M \times S^1$ to M is considered, and b_α represents fluctuation of the metric on $M \times S^1$.

where $M = \ldots + N_{s-1} + N_s + N_{s+1} + \ldots$. Here all n_s are different and must be integers due to the single-valuedness of $V_{[I]}$. From (5.1), we can evaluate the pure-gauge background on $\pi^{-1}(U_{[I]})$ as

$$\begin{aligned}
\hat{A}^{[I]}_\alpha &= -iE^{[I]M}_\alpha \frac{\partial V_{[I]}}{\partial z^M_{[I]}} V^\dagger_{[I]} \\
&= -\frac{1}{R} b^{[I]}_\alpha \,\mathrm{diag}(\ldots, \underbrace{n_{s-1}, \ldots, n_{s-1}}_{N_{s-1}}, \underbrace{n_s, \ldots, n_s}_{N_s}, \underbrace{n_{s+1}, \ldots, n_{s+1}}_{N_{s+1}}, \ldots), \\
\hat{A}_{D+1} &= -iE^{[I]M}_{D+1} \frac{\partial V_{[I]}}{\partial z^M_{[I]}} V^\dagger_{[I]} \\
&= \frac{1}{R} \mathrm{diag}(\ldots, \underbrace{n_{s-1}, \ldots, n_{s-1}}_{N_{s-1}}, \underbrace{n_s, \ldots, n_s}_{N_s}, \underbrace{n_{s+1}, \ldots, n_{s+1}}_{N_{s+1}}, \ldots).
\end{aligned} \tag{5.2}$$

Note that $\hat{A}^{[I]}_\alpha$ is patch-dependent as $b^{[I]}_\alpha$ does. This patch-dependence originates from considering a particular background. If there is overlap between $U_{[I]}$ and $U_{[I']}$, $\hat{A}^{[I]}_\alpha$ is gauge-transformed to $\hat{A}^{[I']}_\alpha$ by

$$\begin{aligned}
V_{[II']} &= V_{[I']} V^\dagger_{[I]} \qquad (5.3)\\
&= \mathrm{diag}(\ldots, \underbrace{e^{-\frac{i}{R} n_{s-1} v_{[II']}}, \ldots, e^{-\frac{i}{R} n_{s-1} v_{[II']}}}_{N_{s-1}}, \underbrace{e^{-\frac{i}{R} n_s v_{[II']}}, \ldots, e^{-\frac{i}{R} n_s v_{[II']}}}_{N_s}, \\
&\qquad\qquad \underbrace{e^{-\frac{i}{R} n_{s+1} v_{[II']}}, \ldots, e^{-\frac{i}{R} n_{s+1} v_{[II']}}}_{N_{s+1}}, \ldots).
\end{aligned}$$

while $\hat{A}_{D+1}$ is invariant. $e^{-\frac{i}{R} v_{[II']}}$ is nothing but the transition function between $U_{[I]}$ and $U_{[I']}$, so that $V_{[II']}$ is well-defined. The background (5.2) is gauge-equivalent to the trivial background $A_A = 0$, which corresponds to the trivial vacuum of the theory.

As in section 3, we denote the fluctuations on $\pi^{-1}(U_{[I]})$ around the background (5.2) by $\tilde{A}^{[I]}_A$, while we continue to use A_A for the gauge fields around the trivial background $A_A = 0$, which are patch-independent. The background (5.2) is gauge-transformed to the trivial background by $V^\dagger_{[I]}$, so that as in (3.5) we have

$$A^{(s,t)}_A = e^{-\frac{i}{R}(n_s - n_t) y_{[I]}} \tilde{A}^{[I](s,t)}_A. \tag{5.4}$$

We also see from (5.3)

$$\tilde{A}^{[I'](s,t)}_A = e^{-\frac{i}{R}(n_s - n_t) v_{[II']}} \tilde{A}^{[I](s,t)}_A. \tag{5.5}$$

We can make the Fourier transformations locally on each patch with respect to $y_{[I]}$. On $\pi^{-1}(U_{[I]})$, $A^{(s,t)}_A$ and $\tilde{A}^{[I](s,t)}_A$ are expanded as

$$\begin{aligned}
A^{(s,t)}_A(x_{[I]}, y_{[I]}) &= \sum_m A^{[I](s,t)}_{A,m}(x_{[I]}) e^{\frac{i}{R} m y_{[I]}}, \\
\tilde{A}^{[I](s,t)}_A(x_{[I]}, y_{[I]}) &= \sum_m \tilde{A}^{[I](s,t)}_{A,m}(x_{[I]}) e^{\frac{i}{R} m y_{[I]}}.
\end{aligned} \tag{5.6}$$

From these equalities, we easily see that

$$\begin{aligned}
A^{[I'](s,t)}_{A,m}(x_{[I']}) &= e^{\frac{i}{R}mv_{[II']}}A^{[I](s,t)}_{A,m}(x_{[I]}),\\
\tilde{A}^{[I'](s,t)}_{A,m}(x_{[I']}) &= e^{\frac{i}{R}(m-(n_s-n_t))v_{[II']}}\tilde{A}^{[I](s,t)}_{A,m}(x_{[I]}).
\end{aligned} \tag{5.7}$$

The relation (5.4) is translated to the relation between the KK modes:

$$A^{[I](s,t)}_{A,m-(n_s-n_t)} = \tilde{A}^{[I](s,t)}_{A,m}. \tag{5.8}$$

This is of course consistent with (5.7). The theory around the trivial background of (4.7) is equivalent to the theory around the background (5.2) of (4.7) under the identification of the KK modes (5.8).

As in the $R^p \times S^1$ case, different consistent truncations of the theory around the trivial vacuum of (4.7) give rise to different theories on the base space. The U(1) isometry indeed ensures that the following truncations are consistent ones. Keeping only $A^{[I](s,t)}_{A,0}$ in the theory around the trivial background of (4.7) generates the theory around the trivial vacuum $a_\alpha = 0,\ \phi = 0$ of (4.10). Keeping only $\tilde{A}^{[I](s,t)}_{A,0}$ in the theory around the background (5.2) is equivalent to keeping only $A^{[I](s,t)}_{A,-(n_s-n_t)}$ in the theory around the trivial background, and generates the theory around a nontrivial background of (4.10), which we will discuss shortly. By taking $M = N \times \infty$, $N_s = N$ and $n_s = s$ and imposing the periodicity $A^{(s,t)}_A = A^{(s-t)}_A$, the T-duality between the theories on the total space and on the base space is achieved in the same way as the $R^p \times S^1$ case.

It is seen from (4.8) and (5.2) that keeping only $A^{[I](s,t)}_{A,-(n_s-n_t)}$ results in the theory around a background of (4.10),

$$\begin{aligned}
\hat{a}^{[I]}_\alpha &= -b^{[I]}_\alpha \hat{\phi},\\
\hat{\phi} &= 2\pi\tilde{R}\,\mathrm{diag}(\ldots,\underbrace{n_{s-1},\ldots,n_{s-1}}_{N_{s-1}},\underbrace{n_s,\ldots,n_s}_{N_s},\underbrace{n_{s+1},\ldots,n_{s+1}}_{N_{s+1}},\ldots),
\end{aligned} \tag{5.9}$$

where $\tilde{R} = \frac{1}{2\pi R}$. It is remarkable that the gauge fields take the monopole-like configuration described by $b^{[I]}_\alpha$. We discuss the quantization of the fluxes in section 6. The background (5.9) would correspond to a vacuum of (4.10), because the background (5.2) corresponds to a vacuum of (4.7). Indeed it satisfies the equations

$$\begin{aligned}
\hat{f}_{\alpha\beta} + b_{\alpha\beta}\hat{\phi} &= 0,\\
e^\mu_\alpha \partial_\mu \hat{\phi} - i[\hat{a}_\alpha, \hat{\phi}] &= 0,
\end{aligned} \tag{5.10}$$

which give the conditions for the vacua.

If there is overlap between $U_{[I]}$ and $U_{[I']}$, the gauge fields and the Higgs field in $U_{[I]}$ and $U_{[I']}$ are related by the gauge transformation

$$\begin{aligned}
\hat{a}^{[I']}_\alpha &= -ie^{[I]\mu}_\alpha \partial^{[I]}_\mu V_{[II']}V^\dagger_{[II']} + V_{[II']}\hat{a}^{[I]}_\alpha V^\dagger_{[II']},\\
\hat{\phi} &= V_{[II']}\hat{\phi}V^\dagger_{[II']}.
\end{aligned} \tag{5.11}$$

We denote the (s,t) block of fluctuations around (5.9) on $U_{[I]}$ by $\tilde{a}_\alpha^{[I](s,t)}$ and $\tilde{\phi}^{[I](s,t)}$, which are identified with $\tilde{A}^{[I](s,t)}_{\alpha,0} = A^{[I](s,t)}_{\alpha,-(n_s-n_t)}$ and $\tilde{A}^{[I](s,t)}_{D+1,0} = A^{[I](s,t)}_{D+1,-(n_s-n_t)}$, respectively. The fluctuations are gauge-transformed from $U_{[I]}$ to $U_{[I']}$ as

$$\begin{aligned}\tilde{a}_\alpha^{[I'](s,t)} &= e^{-\frac{i}{R}(n_s-n_t)v_{[II']}}\tilde{a}_\alpha^{[I](s,t)},\\ \tilde{\phi}^{[I'](s,t)} &= e^{-\frac{i}{R}(n_s-n_t)v_{[II']}}\tilde{\phi}^{[I](s,t)}.\end{aligned} \tag{5.12}$$

For completeness, we state explicitly the T-duality in this case: the theory around (5.9) of (4.10) with $M = N\times\infty$, $N_s = N$, $n_s = s$ and the periodicity condition $\tilde{a}_\alpha^{[I](s,t)} = \tilde{a}_\alpha^{[I](s-t)}$ and $\tilde{\phi}^{[I](s,t)} = \tilde{\phi}^{[I](s-t)}$ is equivalent to the theory around the trivial vacuum of (4.7) with the gauge group $\mathrm{U}(N)$ and the overall factor $\sum_w$. The relation between the fields on the total space and on the base space is given by

$$\begin{aligned}A_\alpha(x_{[I]}, y_{[I]}) &= \sum_w \tilde{a}_\alpha^{[I](w)}(x_{[I]})e^{-\frac{i}{R}wy_{[I]}},\\ A_{D+1}(x_{[I]}, y_{[I]}) &= \sum_w \tilde{\phi}^{[I](w)}(x_{[I]})e^{-\frac{i}{R}wy_{[I]}}.\end{aligned} \tag{5.13}$$

In order that the fields in the lefthand sides in (5.13) are the ones around the trivial vacuum of (4.7), they must be patch-independent. It is seen from (5.12) they are indeed patch-independent. Interestingly, the monopole-like charges are identified with the momenta. It is indeed easy to check explicitly that the Fourier transformation (5.13) realizes the T-duality, as we did in section 2.

6. Nontrivial vacua on total space

So far we have been concerned with the theory around the trivial vacuum on the total space. In general, there are nontrivial vacua on the total space. In this section, we discuss the nontrivial vacua on the total space and their relation to the vacua on the base space.

First, we classify the vacua on the total space. Let the gauge group of (4.7) be $\mathrm{U}(M)$. The vacua of (4.7) are given by the space of the flat connections modulo the gauge transformations, which are parameterized by the holonomies (the Wilson lines) along the nontrivial generators of the fundamental group. Let us consider the closed loop along the fiber S^1. The Wilson line along the loop for the flat connections is diagonalized as [22, 23]

$$\begin{aligned}W &= P\exp\left(i\int_0^{2\pi R} dy_{[I]}A_y(x_{[I]}, y_{[I]})\right)\\ &= \mathrm{diag}(\underbrace{e^{2\pi i\theta^{(1)}},\ldots,e^{2\pi i\theta^{(1)}}}_{M^{(1)}},\ldots,\underbrace{e^{2\pi i\theta^{(T)}},\ldots,e^{2\pi i\theta^{(T)}}}_{M^{(T)}}),\end{aligned} \tag{6.1}$$

where $M = M^{(1)} + \ldots + M^{(T)}$, and all $\theta^{(a)}$ are constants different each other and satisfying $0 \le \theta^{(a)} < 1$. If the loop is contractable, $W = \mathbf{1}_M$, namely $T = 1$ and $\theta^{(1)} = 0$. In the case of the nontrivial fiber bundles, $\theta^{(a)}$ are in general discretized, as we will see shortly. One can take a gauge in which A_y is diagonal and constant:

$$\hat{A}_y = \frac{1}{R}\mathrm{diag}(\underbrace{\theta^{(1)},\ldots,\theta^{(1)}}_{M^{(1)}},\ldots,\underbrace{\theta^{(T)},\ldots,\theta^{(T)}}_{M^{(T)}}), \tag{6.2}$$

which gives (6.1). By solving the flatness condition $F^{[I]}_{\mu y}=0$, one finds that $A^{[I]}_{\mu}$ must have the same block structure as $\hat{A}_y$ and be $y_{[I]}$-independent:

$$\hat{A}^{[I]}_{\mu}(x_{[I]})=\begin{pmatrix}\hat{A}^{[I](1)}_{\mu}(x_{[I]}) & & \\ & \ddots & \\ & & \hat{A}^{[I](T)}_{\mu}(x_{[I]})\end{pmatrix}, \tag{6.3}$$

where the diagonal block component, $\hat{A}^{[I](a)}_{\mu}$, is an $M^{(a)}\times M^{(a)}$ matrix and all the off-diagonal block components vanish. $\hat{A}^{[I](a)}_{\mu}$ are determined by the flatness condition $F^{[I]}_{\mu\nu}=0$, up to the gauge transformations that are elements of $\mathrm{U}(M^{(1)})\times\ldots\times\mathrm{U}(M^{(T)})$ and $y_{[I]}$-independent. The vacua on the total space are parameterized by $M^{(a)}$, $\theta^{(a)}$ and $\hat{A}^{[I](a)}_{\mu}$ modulo the gauge transformations.

Next, we examine the relation between the vacua on the total space and the base space. Each vacuum of (4.10) is lifted up to a vacuum of (4.7). On the other hand, the configurations given by (6.2) and (6.3) are $y_{[I]}$-independent, so that they correspond to the vacua on the base space. This implies that the map from the space of the vacua on the base space to those on the total space is surjective. However, it is not injective. Suppose that $\hat{A}^{[I](a)}_{\mu}$ can be block-diagonalized as

$$\hat{A}^{[I](a)}_{\mu}=\begin{pmatrix}\ddots & & & \\ & \hat{A}^{[I](a;s-1)}_{\mu} & & & \\ & & \hat{A}^{[I](a;s)}_{\mu} & & \\ & & & \hat{A}^{[I](a;s+1)}_{\mu} & \\ & & & & \ddots\end{pmatrix}, \tag{6.4}$$

where $\hat{A}^{[I](a;s)}_{\mu}$ is an $N^{(a)}_s\times N^{(a)}_s$ matrix and $M^{(a)}=\ldots+N^{(a)}_{s-1}+N^{(a)}_s+N^{(a)}_{s+1}+\ldots$. Then, by applying the gauge transformation of the type $V_{[I]}$, one can shift $(\theta^{(a)},\ldots,\theta^{(a)})$ in $\hat{A}_y$ as

$$\begin{aligned}(\underbrace{\theta^{(a)},\ldots,\theta^{(a)}}_{M^{(a)}})\to{}&(\underbrace{\theta^{(a)},\ldots,\theta^{(a)}}_{M^{(a)}})\\ &+(\ldots,\underbrace{n^{(a)}_{s-1},\ldots,n^{(a)}_{s-1}}_{N^{(a)}_{s-1}},\underbrace{n^{(a)}_{s},\ldots,n^{(a)}_{s}}_{N^{(a)}_{s}},\underbrace{n^{(a)}_{s+1},\ldots,n^{(a)}_{s+1}}_{N^{(a)}_{s+1}},\ldots),\end{aligned} \tag{6.5}$$

where $n^{(a)}_s$ can be different. We denote this shifted $\hat{A}_y$ by $\hat{A}'_y$. The gauge-transformed configuration described by $\hat{A}'_y$ represents the same vacuum on the total space as the original configuration described by $\hat{A}_y$. As in the case of the trivial vacuum on the total space, due to the variety of the choices of $n^{(a)}_s$, one can consistently truncate the theory around the vacuum of (4.7) described by (6.2) and (6.3) in different ways to obtain different theories on the base space. Those theories on the base space are the ones around the vacua of (4.10) given by

$$\begin{aligned}\hat{a}^{[I](a;s)}_{\alpha}&=-\frac{1}{R}(\theta^{(a)}+n^{(a)}_s)b^{[I]}_{\alpha}\mathbf{1}_{N^{(a)}_s}+e^{[I]\mu}_{\alpha}\hat{A}^{[I](a;s)}_{\mu},\\ \hat{\phi}&=\hat{A}'_y.\end{aligned} \tag{6.6}$$

Indeed, a general solution to the vacuum condition (5.10) takes this form in the gauge in which ϕ is diagonal. It is seen from the first equation in (5.10) that $e^{[I]\mu}_{\alpha}\hat{A}^{[I](a;s)}_{\mu}$ gives a flat connection on the base space. It is easy to see that the T-duality also holds for the theories around the nontrivial vacua on the total space. In fact, by making s in each block run from $-\infty$ to ∞, taking $N_s^{(a)} = N^{(a)}$, $M^{(a)} = N^{(a)} \times \infty$ and $n_s^{(a)} = s$ and imposing the periodicity condition on the fluctuations around (6.6), one obtains the theory around the vacuum of (4.7) described by (6.2) and (6.3) with $M^{(a)}$ replaced by $N^{(a)}$.

Finally, we comment on the quantization of the fluxes. For the vacua (6.6), the first equation in (5.10) implies

$$\hat{f}^{(a;s)}_{\alpha\beta} = -\frac{1}{R}(\theta^{(a)} + n_s^{(a)}) b_{\alpha\beta} \mathbf{1}_{N_s^{(a)}} \tag{6.7}$$

The 1st Chern class evaluated from both sides of (6.7) is an element of the 2nd cohomology class with the integer coefficients of the base manifold, $H^2(B, Z)$. If the fiber bundle is nontrivial, the curvature of the principal S^1 bundle, $\frac{1}{R}b_{\alpha\beta}$, is a nontrivial element of $H^2(B, Z)$ so that we have a relation $p(\theta^{(a)} + n_s^{(a)}) \in Z$ with a certain $p \in Z_+$. This is the quantization of the fluxes. From this relation, we can deduce

$$\theta^{(a)} = \frac{l^{(a)}}{p}, \tag{6.8}$$

where $l^{(a)}$ are integers satisfying $0 \leq l^{(a)} \leq p-1$. If the fiber bundle is trivial, the curvature is a unit element of $H^2(B, Z)$ so that we have no relation for $\theta^{(a)} + n_s^{(a)}$. There is no quantization of the fluxes in this case, and $\theta^{(a)}$ take continuous values. For example, in U(M) Yang Mills on $R^p \times S^1$, the vacua are completely parameterized by $M^{(a)}$ and $\theta^{(a)}$ [22, 23]. Here $\theta^{(a)}$ are continuous parameters and $\hat{A}_\mu = 0$. The value of p should also be determined from the structure of the fundamental group on the total space, because the vacua on the total space are given by the space of the flat connections modulo the gauge transformations and the flat connections are classified by the holonomies that are a representation of the fundamental group. One can see in the next section that this is indeed the case in some examples. Note that in the $R^p \times S^1$ case, $\pi_1(R^p \times S^1) = \pi_1(S^1) = Z$ so that there is no quantization of $\theta^{(a)}$.

7. Examples

In this section, we present some examples of the T-duality for gauge theory on curved space described as a principal S^1 bundle. In sections 3.1, 3.2 and 3.3, we treat S^3 and S^3/Z_k as S^1 on S^2, S^5 and S^5/Z_k as S^1 on CP^2 and the Heisenberg nilmanifold as S^1 on T^2, respectively.

7.1 S^3 and S^3/Z_k as S^1 on S^2

We consider S^3 with radius 2 and regard it as the U(1) Hopf bundle on S^2 with radius 1. S^3 with radius 2 is defined by

$$\{(w_1, w_2) \in C^2 \mid |w_1|^2 + |w_2|^2 = 4\}. \tag{7.1}$$

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The Hopf map $\pi:\ S^3 \to CP^1\ (S^2)$ is defined by

$$(w_1, w_2) \to [(w_1, w_2)] \equiv \{\lambda(w_1, w_2) | \lambda \in C\backslash\{0\}\}. \tag{7.2}$$

Two patches are introduced on CP^1: the patch 1 ($w_1 \neq 0$) and the patch 2 ($w_2 \neq 0$). On the patch 1 the local trivialization is given by

$$(w_1, w_2) \to \left(\frac{w_2}{w_1}, \frac{w_1}{|w_1|}\right), \tag{7.3}$$

where $\frac{w_2}{w_1}$ is the local coordinate of CP^1, while on the patch 2 the local trivialization is given by

$$(w_1, w_2) \to \left(\frac{w_1}{w_2}, \frac{w_2}{|w_2|}\right), \tag{7.4}$$

where $\frac{w_1}{w_2}$ is the local coordinate of CP^1.

The equation (7.1) is solved as

$$w_1 = 2\cos\frac{\theta}{2}\, e^{i\sigma_1}, \quad w_2 = 2\sin\frac{\theta}{2}\, e^{i\sigma_2}, \tag{7.5}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \sigma_1,\ \sigma_2 < 2\pi$. We put

$$\varphi = \sigma_1 - \sigma_2, \quad \psi = \sigma_1 + \sigma_2, \tag{7.6}$$

and can change the ranges of φ and ψ to $0 \leq \varphi < 2\pi$ and $0 \leq \psi < 4\pi$. The periodicity is expressed as

$$(\theta, \varphi, \psi) \sim (\theta, \varphi + 2\pi, \psi + 2\pi) \sim (\theta, \varphi, \psi + 4\pi). \tag{7.7}$$

From the local trivializations (7.3) and (7.4), one can see that θ and ϕ are regarded as the angular coordinates of the base space S^2 through the stereographic projection. The patch 1 corresponds to $0 \leq \theta < \pi$, while the patch 2 corresponds to $0 < \theta \leq \pi$. The metric of S^3 is given as follows:

$$\begin{aligned} ds^2_{S^3} &= |dw_1|^2 + |dw_2|^2 \\ &= d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\varphi)^2. \end{aligned} \tag{7.8}$$

The correspondence to the notation of section 4 is as follows:

$$\begin{aligned} z^M_{[1]} &= (\theta, \varphi, \psi + \varphi), & x^\mu_{[1]} &= (\theta, \varphi), & y_{[1]} &= \psi + \varphi, \\ z^M_{[2]} &= (\theta, \varphi, \psi - \varphi), & x^\mu_{[2]} &= (\theta, \varphi), & y_{[2]} &= \psi - \varphi, \\ b^{[1]}_\theta &= 0, & b^{[1]}_\varphi &= \cos\theta - 1, \\ b^{[2]}_\theta &= 0, & b^{[2]}_\varphi &= \cos\theta + 1, \\ R &= 2. \end{aligned} \tag{7.9}$$

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The metric and the zweibeins of the base space S^2 are given by

$$ds^2_{S^2} = d\theta^2 + \sin^2\theta d\varphi^2,$$
$$e^1_\theta = 1, \qquad e^2_\varphi = \sin\theta. \tag{7.10}$$

(4.10) takes the form

$$S_{S^2} = \frac{1}{g^2_{S^2}} \int d\theta d\varphi \sin\theta \mathrm{Tr}\left(\frac{1}{2}(f_{12}-\phi)^2 + \frac{1}{2}(D_1\phi)^2 + \frac{1}{2}(D_2\phi)^2\right). \tag{7.11}$$

The vacua of (7.11) takes the form

$$\begin{aligned}
\hat{a}_1^{[1],[2]} &= 0,\\
\hat{a}_2^{[1]} &= \tan\frac{\theta}{2}\,\hat{\phi},\\
\hat{a}_2^{[2]} &= -\cot\frac{\theta}{2}\,\hat{\phi},\\
\hat{\phi} &= \frac{1}{2}\mathrm{diag}(\ldots, n_{i-1}, n_i, n_{i+1}, \ldots).
\end{aligned} \tag{7.12}$$

The configuration of the i-th diagonal element of the gauge fields are the Dirac monopole with the monopole charge $n_i/2$. That n_i are integers is consistent with Dirac's quantization condition. The vacuum of Yang Mills on S^3 is unique due to $\pi_1(S^3) = 0$. There are no degrees of freedom corresponding to $\theta^{(a)}$ and $\hat{A}^{[I]}_\mu$, and $p = 1$. The value of p is also determined consistently by

$$W = P\exp\left(i\int_0^{4\pi} dy_{[I]}\hat{A}_y\right) = 1. \tag{7.13}$$

The theories around all the vacua (7.12) originate from the theory around the trivial vacuum on S^3.

As shown in the previous section in general, there holds the T-duality between the original $\mathrm{U}(N)$ Yang Mills on S^3 and (7.11) with the gauge group $\mathrm{U}(N\times\infty)$ and the periodicity condition imposed.

The relationship between the gauge fields on S^3 and the gauge fields and the Higgs field on S^2 is given in (5.13) with $\alpha = 1, 2$.

The gauge fields on S^3 are expanded in terms of the vector spherical harmonics on S^3, while the gauge fields and the Higgs field on S^2 in this case are expanded together in terms of the vector monopole harmonics [24, 25]. From (5.13), one can read off the relationship between the spherical harmonics on S^3 and the monopole harmonics, which was found in [12, 10] to show the T-duality between $\mathcal{N} = 4$ SYM on $R\times S^3(/Z_k)$ and 2+1 SYM on $R\times S^2$.

The lens space S^3/Z_k ($k \in Z_+$) is defined by introducing an identification, $(w_1, w_2) \sim (w_1\, e^{\frac{2\pi i}{k}}, w_2\, e^{\frac{2\pi i}{k}})$ into the definition of S^3 and can also be regarded as S^1 on S^2. The local trivialization on the patch 1 is

$$(w_1, w_2) \to \left(\frac{w_2}{w_1}, \left(\frac{w_1}{|w_1|}\right)^k\right), \tag{7.14}$$

while the local trivialization on the patch 2 is

$$(w_1, w_2) \to \left(\frac{w_1}{w_2}, \left(\frac{w_2}{|w_2|}\right)^k\right). \tag{7.15}$$

The radius of the fiber is replaced with $R = \frac{2}{k}$. The form of the action on the base space is the same as (7.11). The counterpart of (7.12) is obtained by replacing $\hat{\phi}$ in (7.12) with $\hat{\phi} = \frac{1}{2}\mathrm{diag}(\ldots, \beta_{i-1} + kn_{i-1}, \beta_i + kn_i, \beta_{i+1} + kn_{i+1}, \ldots)$, where β_i are integers and $0 \leq \beta_i \leq k-1$. The values and the multiplicities of β_i label the vacua on S^3/Z_k and correspond to $\theta^{(a)}$ and $M^{(a)}$, respectively. The vacua on S^3/Z_k are classified by the holonomy along the generator of $\pi_1(S^3/Z_k) = Z_k$, which can be evaluated from

$$W = P\exp\left(i\int_0^{4\pi/k} dy_{[I]}\hat{A}_y\right). \tag{7.16}$$

The value of p is determined by $W^k = 1$ as $p = k$.

7.2 S^5 and S^5/Z_k as S^1 on CP^2

We regard S^5 as a U(1) bundle on CP^2 as follows. S^5 with the radius 1 is defined by

$$\{(w_1, w_2, w_3) \in C^3 \mid |w_1|^2 + |w_2|^2 + |w_3|^2 = 1\}. \tag{7.17}$$

The Hopf map $\pi: S^5 \to CP^2$ is given by

$$(w_1, w_2, w_3) \to [(w_1, w_2, w_3)] \equiv \{\lambda(w_1, w_2, w_3) | \lambda \in C\backslash\{0\}\}. \tag{7.18}$$

Three patches are introduced on CP^2: the patch 1 ($w_1 \neq 0$), the patch 2 ($w_2 \neq 0$) and the patch 3 ($w_3 \neq 0$). The fiber on the patch I given by the local trivialization is parameterized by $\frac{w_I}{|w_I|}$. (7.17) is solved as

$$\begin{aligned} w_1 &= \cos\chi\, e^{i\tau}, \\ w_2 &= \sin\chi\cos\frac{\theta}{2}\, e^{i(\tau + \frac{\psi+\varphi}{2})}, \\ w_3 &= \sin\chi\sin\frac{\theta}{2}\, e^{i(\tau + \frac{\psi-\varphi}{2})}. \end{aligned} \tag{7.19}$$

where $0 \leq \tau < 2\pi$, $0 \leq \chi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$ and $0 \leq \psi < 4\pi$. The periodicity in this case is given by

$$(\chi, \theta, \varphi, \psi, \tau) \sim (\chi, \theta, \varphi, \psi, \tau + 2\pi) \sim (\chi, \theta, \varphi, \psi + 4\pi, \tau) \sim (\chi, \theta, \varphi + 2\pi, \psi + 2\pi, \tau). \tag{7.20}$$

The metric of S^5 is given by

$$\begin{aligned} ds^2_{S^5} &= |dw_1|^2 + |dw_2|^2 + |dw_3|^2 \\ &= ds^2_{CP^2} + \omega^2. \end{aligned} \tag{7.21}$$

with

$$
\begin{aligned}
ds^2_{CP^2} &= d\chi^2 + \frac{1}{4}\sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\chi(d\psi + \cos\theta d\varphi)^2), \\
\omega &= d\tau + \frac{1}{2}\sin^2\chi(d\psi + \cos\theta d\varphi),
\end{aligned}
\tag{7.22}
$$

where $ds^2_{CP^2}$ is the metric of CP^2, which is called the Fubini-Study metric while ω is the connection 1-form. The correspondence to the notation of section 4 is

$$
\begin{aligned}
z^M_{[1]} &= (\chi,\theta,\varphi,\psi,\tau), & x^\mu_{[1]} &= (\chi,\theta,\varphi,\psi), & y_{[1]} &= \tau, \\
z^M_{[2]} &= (\chi,\theta,\varphi,\psi,\tau+\frac{1}{2}(\psi+\varphi)), & x^\mu_{[2]} &= (\chi,\theta,\varphi,\psi), & y_{[2]} &= \tau+\frac{1}{2}(\psi+\varphi), \\
z^M_{[3]} &= (\chi,\theta,\varphi,\psi,\tau+\frac{1}{2}(\psi-\varphi)), & x^\mu_{[3]} &= (\chi,\theta,\varphi,\psi), & y_{[3]} &= \tau+\frac{1}{2}(\psi-\varphi), \\
b^{[1],[2],[3]}_\chi &= 0, & b^{[1],[2],[3]}_\theta &= 0, \\
b^{[1]}_\varphi &= \frac{1}{2}\sin^2\chi\cos\theta, & b^{[1]}_\psi &= \frac{1}{2}\sin^2\chi, \\
b^{[2]}_\varphi &= \frac{1}{2}(\sin^2\chi\cos\theta - 1), & b^{[2]}_\psi &= \frac{1}{2}(\sin^2\chi - 1), \\
b^{[3]}_\varphi &= \frac{1}{2}(\sin^2\chi\cos\theta + 1), & b^{[3]}_\psi &= \frac{1}{2}(\sin^2\chi - 1), \\
R &= 1.
\end{aligned}
\tag{7.23}
$$

$b^{[I]}_\mu$ in (7.23) is called the gravitational and electromagnetic instanton in [26]. The vacuum on S^5 is unique. The vacua on CP^2 are given by (5.9). The value of p is determined by $W = P\exp\left(i\int_0^{2\pi} dy_{[I]}A_y\right) = 1$ as $p = 1$.

The lens space S^5/Z_k is treated in the same way as S^3/Z_k.

7.3 Heisenberg nilmanifold as S^1 on T^2

The Heisenberg nilmanifold [27, 28] is a twisted 3-torus that has the following periodicity condition.

$$
(x^1,x^2,y) \sim (x^1,x^2+L_2,y) \sim (x^1,x^2,y+L_y) \sim (x^1+L_1,x^2,y-\kappa L_1 x^2), \tag{7.24}
$$

where x^1, x^2 and y are the coordinates of the nilmanifold, and κ is determined from consistency of (7.24) as $\kappa = l\frac{L_y}{L_1L_2}, (l \in Z)$. The metric of the nilmanifold is given by

$$
ds^2 = (dx^1)^2 + (dx^2)^2 + (dy + \kappa x^1 dx^2)^2. \tag{7.25}
$$

We regard the nilmanifold as a U(1) bundle on T^2 parameterized by x^1 and x^2. We need two patches on T^2 to describe the U(1) bundle. We define the patch 1 as the region $0 < x^1_{[1]} < L_1$, and the patch 2 as the region $-\frac{L_1}{2} < x^1_{[2]} < \frac{L_1}{2}$. On each patch, the nilmanifold is locally trivialized such that it is parameterized by $(x^1_{[1]}, x^2_{[1]}, y_{[1]})$ on the patch 1 and $(x^1_{[2]}, x^2_{[2]}, y_{[2]})$ on the patch 2, where $(x^1_{[I]}, x^2_{[I]})$ are the local coordinate of the base manifold T^2 and $y_{[I]}$ parameterizes the S^1 fiber direction. On the overlap between the two patches, the transition functions are given as follows:

$$
x^1_{[2]} = x^1_{[1]} \ , x^2_{[2]} = x^2_{[1]}, \ \ y_{[2]} = y_{[1]}, \tag{7.26}
$$

in the region $0 < x^1_{[1]} < \frac{L_1}{2}$, $0 < x^1_{[2]} < \frac{L_1}{2}$, and

$$x^1_{[2]} = x^1_{[1]} - L_1,\ \ x^2_{[2]} = x^2_{[1]},\ \ y_{[2]} = y_{[1]} - \kappa L_1 x^2_{[1]}, \tag{7.27}$$

in the region $\frac{L_1}{2} < x^1_{[1]} < L_1$, $-\frac{L_1}{2} < x^1_{[2]} < 0$. Note that this representation of the nilmanifold in terms of the patches is equivalent to the definition in terms of the periodicity condition (7.24).

The correspondence to the notation of section 4 is as follows:

$$\begin{aligned} b_1^{[I]} &= 0, \quad b_2^{[I]} = \kappa x^1_{[I]},\ \ I = 1,2, \\ b_{12} &= \kappa, \\ R &= \frac{L_y}{2\pi}. \end{aligned} \tag{7.28}$$

$b_\alpha^{[I]}$ represents the constant magnetic flux with the strength l on T^2. This implies $p = l$. The value of p is also determined from the structure of the fundamental group of the Heisenberg nilmanifold. As discussed in [27], W^l equals the Wilson line along an element of the commutator subgroup of the fundamental group. For the U(1) part of the a-th block of the gauge fields, therefore, we have

$$\left(P \exp\left(i\int_0^{L_y} dy_{[I]} \hat{A}_y^{(a)}\right)\right)^l = 1, \tag{7.29}$$

from which $p = l$ follows. In this case, $e_\alpha^{[I]\mu}\hat{A}_\mu^{[I](a;s)}$ in (6.6) can give nontrivial Wilson lines along the generators of the fundamental group of T^2 and contribute to the classification of the vacua. (4.10) takes the form

$$S = \frac{1}{g_{T^2}}\int dx_1 dx_2 \mathrm{Tr}\left(\frac{1}{2}(f_{12}+\kappa\phi)^2 + \frac{1}{2}(D_1\phi)^2 + \frac{1}{2}(D_2\phi)^2\right). \tag{7.30}$$

8. Conclusion and discussion

In this paper, we first discussed the variety of the consistent truncations from Yang Mills on the total space to Yang Mills with the Higgs field on the base space in the trivial and nontrivial principal S^1 bundles. Different consistent truncations of the theory around a vacuum of Yang Mills on the total space yield the theories around different nontrivial vacua of Yang Mills with the Higgs field on the base space. In the case of the nontrivial S^1 bundles, the nontrivial vacua on the base space have monopole-like gauge configurations. By using this viewpoint, we showed the T-duality between the theories on the total space and the base space in the nontrivial bundle case as well as the trivial bundle case. The difference between these two cases is that in the nontrivial bundle case, the vacuum configurations of the gauge fields are the monopole-like ones and the Fourier transformation must be made locally on each patch. It is remarkable that the monopole-like charges are identified with the momenta on the total space. We also classified the vacua on the total space and their relation to the vacua on the base space. The quantization of the monopole-like charges on the base space is understood from the structure of the fundamental group on the total space.

It is easy to add adjoint matters to Yang Mills on the total space and to introduce supersymmetry. In [10], we showed the T-duality between the theory around the trivial vacuum of $\mathcal{N}=4$ SYM on $R\times S^3/Z_k$ and the theory around a vacuum of 2+1 SYM on $R\times S^2$. Our results in this paper show that the T-duality also holds for the nontrivial vacua of $\mathcal{N}=4$ SYM on $R\times S^3/Z_k$.

In this paper, we restricted ourselves to principal U(1) bundles. Nonabelian generalization is important. Typical examples are S^7 as S^3 (SU(2)) on S^4, SU(3) as U(2) on CP^2 and so on. In [10], we showed that the theory around each vacuum of Yang Mills with the Higgs on S^2 is equivalent to the theory around a vacuum described by fuzzy spheres of a matrix model. This means that Yang Mills on S^3 and S^3/Z_k is realized in the matrix model. It is interesting to examine what condition is needed in order for a gauge theory on a fiber bundle to be realized in a matrix model. Finally, we expect to apply our findings to (flux) compactification in string theory.

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